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A_X-OPERATOR ON COMPLETE RIEMANNIAN MANIFOLDS

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ABSTRACT
In this paper we give a generalisation of Kostant's Theorem about the A_X-operator associated to a Killing vector field X on a compact Riemannian manifold. Kostant proved (see [6], [5] or [7]) that in a compact Riemannian manifold, the (1, 1) skew-symmetric operator $A_X = L_X - \nabla_X$ associated to a Killing vector field X lies in the holonomy algebra at each point. We prove that in a complete non-compact Riemannian manifold (M, g) the A_X-operator associated to a Killing vector field, with finite global norm, lies in the holonomy algebra at each point. Finally we give examples of Killing vector fields with infinite global norms on non-flat manifolds such that A_X does not lie in the holonomy algebra at any point.

§1. Preliminary results

Let (M, g) be a complete non-compact Riemannian manifold. One can decompose the Lie algebra of skew-symmetric endomorphisms $E(x)$ at a point $x \in M$ in the form $E(x) = G(x) \oplus G(x)^{\perp}$, where $G(x)$ is the holonomy algebra at x and $G(x)^{\perp}$ its orthogonal complement with respect to the local scalar product for tensors of type (1, 1).

Since A_X is skew-symmetric for each Killing vector field X, then at any point $x \in M$ we have:

$$(A_X)_x = (S_X)_x + (B_X)_x, \qquad \text{with } (S_X)_x \in G(x) \quad \text{and} \quad (B_X)_x \in G(x)^{\perp}.$$

Notice that if $B_X = 0$ at some point, the B_X vanishes everywhere, because B_X is parallel.

For two tensor fields T, S of type $(0, s)$ on M, we denote the local scalar product of T and S by $\langle T, S \rangle$, i.e.

$$\langle T, S \rangle = \frac{1}{s!} T_{i_1 \cdots i_s} S^{i_1 \cdots i_s}$$

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and the global scalar product by $\langle\langle T, S\rangle\rangle$,

$$\langle\langle T, S\rangle\rangle = \int_M \langle T, S\rangle \,\mathrm{vol}.$$

We denote by $\Lambda^s(M)$ (resp. $\Lambda_0^s(M)$) the space of s-forms on M (resp. with compact support). Let $L_2^s(M)$ be the completion of $\Lambda_0^s(M)$ with respect to $\langle\langle\ ,\ \rangle\rangle$. We say that a vector field X has a finite global norm (see [8]), if the 1-form associated to X via g, ξ, lies in $L_2^1(M) \cap \Lambda^1(M)$.

To prove our result we have to use Stokes' Theorem and to do this we construct a family of cut-off functions whose supports exhaust M (see [1], [4] or [8]). Let 0 be a fixed point of M. For each $p \in M$ we denote by $\rho(p)$ the geodesic distance from 0 to p. Let $B(\alpha) = \{p \in M \mid \rho(p) < \alpha\}$, $\alpha > 0$. We choose a C^∞-function μ on $\mathbf{R}$ satisfying:

(i) $0 \leqq \mu \leqq 1$ on $\mathbf{R}$,

(ii) $\mu(t) = 1$ for $t \leqq 1$,

(iii) $\mu(t) = 0$ for $t \geqq 2$.

And we set

$$w_\alpha(p) = \mu(\rho(p)/\alpha): \qquad \alpha \in Z^+.$$

One can see that for every $\alpha \in Z^+$, w_α is Lispchitz, hence almost everywhere differentiable on M and furthermore verifies

$$0 \leqq w_\alpha(p) \leqq 1, \qquad \forall p \in M,$$

$$\mathrm{Supp.}\ w_\alpha \subset B_{2\alpha},$$

$$w_\alpha(p) = 1, \qquad \forall p \in B_\alpha,$$

$$\lim_{\alpha \to \infty} w_\alpha = 1,$$

$$|dw_\alpha| \leqq k/\alpha, \qquad \text{almost everywhere on } M.$$

Then we have

LEMMA 1.1 (cf. [1], [4] or [8]). *There exists a positive number A depending only on μ, such that*

$$\| dw_\alpha \otimes \eta \|_{B(2\alpha)} \leqq (A/\alpha^2) \| \eta \|_{B(2\alpha)},$$

$$\| dw_\alpha \wedge \eta \|_{B(2\alpha)} \leqq (A/\alpha^2) \| \eta \|_{B(2\alpha)},$$

for any $\eta \in \Lambda^s(M)$.

We should notice that for $\eta \in L_2^s(M) \cap \Lambda^s(M)$, $w_\alpha \eta$ lies in $\Lambda_0^s(M)$ and $w_\alpha \eta \to \eta$ $(\alpha \to \infty)$ in the strong sense.

§2. Proof of the main result

Let X be a Killing field on a complete non-compact Riemannian manifold (M, g). Let ξ be the 1-form associated to X via g, i.e. $(Z, \xi) = g(X, Z)$ for any vector field Z.

We consider the vector field $w_\alpha^2 B_X X$, which obviously has compact support. By using the parallelism of B_X we have:

$$\operatorname{div}(w_\alpha^2 B_X X) = \operatorname{trace}(V \to \nabla_V(w_\alpha^2 B_X X)) = (B_X X, dw_\alpha^2) + w_\alpha^2 \operatorname{div}. B_X X$$
$$= (B_X X, dw_\alpha^2) - w_\alpha^2 \operatorname{trace} B_X^2.$$

So

$$\operatorname{div}(w_\alpha^2 B_X X) = (B_X X, dw_\alpha^2) + 2\langle w_\alpha B_X, w_\alpha B_X\rangle, \tag{1}$$

$$(B_X X, dw_\alpha^2) = (B_X X)^i (dw_\alpha^2)_i = (B_X)^i_j X^j (dw_\alpha^2)_i = (B_X)^{ik} g_{kj} X^j (dw_\alpha^2)_i$$
$$= (B_X)^{ik} \xi_k (dw_\alpha^2)_i = 2\langle B_X, dw_\alpha^2 \wedge \xi\rangle,$$

because B_X is skew-symmetric.

As $w_\alpha^2 B_X X$ has compact support contained in $B(2\alpha)$, we have

$$\int_M \operatorname{div}(w_\alpha^2 B_X X) = 0.$$

So by (1)

$$\| w_\alpha B_X \|^2_{B(2\alpha)} = -\langle\langle B_X, dw_\alpha^2 \wedge \xi\rangle\rangle_{B(2\alpha)}. \tag{2}$$

By the Schwartz inequality and Lemma 1.1, we have:

$$|\langle\langle B_X, 2w_\alpha dw_\alpha \wedge \xi\rangle\rangle| \leqq \| w_\alpha B_X \|_{B(2\alpha)} \| 2dw_\alpha \wedge \xi \|_{B(2\alpha)}$$
$$\leqq (\tfrac{1}{2})(\| w_\alpha B_X \|^2_{B(2\alpha)} + 4\| dw_\alpha \wedge \xi \|^2_{B(2\alpha)})$$
$$\leqq (\tfrac{1}{2})\| w_\alpha B_X \|^2_{B(2\alpha)} + (2A/\alpha^2)\| \xi \|^2_{B(2\alpha)}.$$

Now (2) yields

$$\| w_\alpha B_X \|^2_{B(2\alpha)} \leqq (\tfrac{1}{2})\| w_\alpha B_X \|^2_{B(2\alpha)} + (2A/\alpha^2)\| \xi \|^2_{B(2\alpha)},$$
$$\| w_\alpha B_X \|^2_{B(2\alpha)} \leqq (4A/\alpha^2)\| \xi |^2_{B(2\alpha)},$$

so letting $\alpha \to \infty$ we get

$$\lim_{\alpha \to \infty} \| w_\alpha B_X \|^2_{B(2\alpha)} \leqq 0,$$

because ξ has finite global norm, therefore

$$B_X = 0.$$

Thus we have proved:

THEOREM 2.1. *Let (M, g) be a complete non-compact Riemannian manifold. Let X be a Killing vector field, with finite global norm. Then A_X lies in the holonomy algebra at each point of M.*

§3. Examples

In this section we shall give two examples of Killing vector fields, on non-flat manifolds, such that its A_X-operator does not lie in the holonomy algebra at any point.

1st Example

We consider the canonical line bundle $K(P_n(\mathbf{C}))$ and $\pi : K(P_n(\mathbf{C})) \to P_n(\mathbf{C})$, the natural projection.

$K(P_n(\mathbf{C}))$ can be endowed with a structure of irreducible non-compact Kähler manifold with vanishing Ricci tensor as follows (see Calabi [2] and [3]); on the inverse image of each subdomain of inhomogeneous coordinate functions, for instance U_0 with coordinates $(z^1, \ldots, z^n)$, we take the local Kähler potential given by:

$$\psi = \phi \circ \pi + u(t),$$

where ϕ is the local Kähler potential on U_0 for the Fubini–Study metric,

$$\phi = (1/k)\left(1 + k \sum_{i=1}^{n} |z^i|^2\right),$$

$$t = \exp\left(((n+1)/2)k\phi\right) |\xi|^2 \qquad (\xi \text{ is the fibre coordinate}),$$

and

$$u(x) = u_0 + (2n/n+1)(\sqrt[n]{1+cx} - 1) - \sum_{j=1}^{n-1} (2(1-w^j)/n+1) \log\left((\sqrt[n]{1+cx} - w^j)/(1-w^j)\right)$$

$(w = \exp(2\pi i/n))$.

One proves that this manifold is an irreducible complete non-compact Kähler manifold whose holonomy algebra is contained in $\mathrm{su}(n+1)$ (vanishing Ricci tensor).

As ψ depends only on $|z^i|^2$ and $|\xi|^2$, it is easy to see that the vector field

$$X = \sqrt{-1}(\xi(\partial/\partial\xi) - \bar{\xi}(\partial/\partial\bar{\xi}))$$

is an infinitesimal automorphism and as $X(\psi) = 0$, is also Killing.

A simple calculation shows that the (1, 1) operator A_X does not lie in su $(n+1)$ at the points where $\xi = 0$.

Now X, which is defined on $\pi^{-1}(U_0)$, can be globalised by considering similar expressions in the inverse images of the remaining subdomains of inhomogeneous coordinates and we obtain a Killing vector field X (globally defined) such that $B_X \neq 0$ at some points, so $B_X \neq 0$ everywhere.

2nd Example

We consider the example of hyper-Kähler manifold furnished by Calabi in [2] and [3]. Let us take the holomorphic contangent bundle $T^{*\prime}(P_n(\mathbf{C}))$ to $P_n(\mathbf{C})$, π is the natural projection from $T^{*\prime}(P_n(\mathbf{C}))$ onto $P_n(\mathbf{C})$.

For each subdomain of inhomogeneous coordinates, for instance U_0 with coordinates $z^1, \ldots, z^n$, we consider the corresponding coordinates on $\pi^{-1}(U_0)$, $\{z^1, \ldots, z^n; \xi_1, \ldots, \xi_n\}$ (ξ_i are the fibre coordinates). In $\pi^{-1}(U_0)$ we take the local Kähler potential given by:

$$\psi(z^i; \xi_i) = \log(1 + |z|^2) + \sqrt{1+4t} - \log(1 + \sqrt{1+4t}),$$

where

$$t = (1 + |z|^2)(|\xi|^2 + |z \cdot \xi|^2),$$

$$|z|^2 = \sum_{i=1}^{n} |z^i|^2, \qquad |\xi|^2 = \sum_{i=1}^{n} |\xi_i|^2,$$

$$z \cdot \xi = \sum_{i=1}^{n} z^i \cdot \xi_i.$$

Now let us consider the two-form of type (2, 0),

$$H = \sum_{i=1}^{n} dz^i \wedge d\xi_i.$$

One can prove that ψ is the Kähler potential for a metric g which is hyper-Kähler with respect to the given complex structure and H.

By the same argument as in the first example one can see that the vector field

$$X = \sqrt{-1}\left(\sum_{i=1}^{n} z^i(\partial/\partial z^i) - \bar{z}^i(\partial/\partial\bar{z}^i)\right)$$

is Killing and $L_X H \neq 0$.

So X is not an infinitesimal automorphism associated to H via g, so $B_X \neq 0$. Indeed if $B_X = 0$, then denoting by J the complex structure corresponding to H, we have

$$L_X J = [A_X, J] = [S_X, J] = 0,$$

because J lies in the centraliser of the holonomy algebra at each point.

One can extend X to the inverse image of the remaining subdomains of inhomogeneous coordinates in a similar way as in the first example, obtaining a Killing vector field (globally defined) X, such that $B_X \neq 0$.

With these two examples, our goal was to remark that one can find Killing vector fields whose A_X-operators do not lie in the holonomy algebra in non-flat manifolds.

For examples of Killing vector fields with finite global norm on manifolds with volume either finite or infinite, see [9].

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